

## Math 3450 - Homework # 2 - Part A

### Equivalence Relations

- A set  $S$  and a relation  $\sim$  on  $S$  is given. For each example, check if  $\sim$  is (i) reflexive, (ii) symmetric, and/or (iii) transitive. If  $\sim$  satisfies the property that you are checking, then prove it. If  $\sim$  does not satisfy the property that you are checking, then give an example to show it.
  - $S = \mathbb{R}$  where  $a \sim b$  if and only if  $a \leq b$ .
  - $S = \mathbb{R}$  where  $a \sim b$  if and only if  $|a| = |b|$ .
  - $S = \mathbb{Z}$  where  $a \sim b$  if and only if  $a|b$ .
  - $S$  is the set of subsets of  $\mathbb{N}$  where  $A \sim B$  if and only if  $A \subseteq B$ . Some examples of elements of  $S$  are  $\{1, 10, 199\}$ ,  $\{2, 7, 10\}$ , and  $\{2, 10, 3, 7\}$ . Note that  $\{2, 7, 10\} \sim \{2, 10, 3, 7\}$
- Consider the set  $S = \mathbb{R}$  where  $x \sim y$  if and only if  $x^2 = y^2$ .
  - Find all the numbers that are related to  $x = 1$ . Repeat this exercise for  $x = \sqrt{2}$  and  $x = 0$ .
  - Prove that  $\sim$  is an equivalence relation on  $S$ .
  - Draw a number line. Draw a picture of the equivalence class of 1. Repeat this for  $x = 0$ ,  $x = \sqrt{6}$ ,  $x = -3$ .
  - Describe the elements of  $S/\sim$ .
- Consider the set  $S = \mathbb{Z}$  where  $x \sim y$  if and only if  $2|(x + y)$ .
  - List six numbers that are related to  $x = 4$ .
  - Prove that  $\sim$  is an equivalence relation on  $S$ .
  - Draw a picture of the set of integers. Next, circle the numbers that are in the equivalence class of  $-3$ .
  - Describe the elements of  $S/\sim$ . Draw a picture of several equivalence classes.
- (Constructing the rational numbers from the integers) Let  $S = \mathbb{Z} \times (\mathbb{Z} - \{0\})$ . Define the relation  $\sim$  on  $S$  where  $(a, b) \sim (c, d)$  if and only if  $ad = bc$ .

- (a) Is  $(1, 5) \sim (-3, -15)$  ?
- (b) Is  $(-1, 1) \sim (2, 3)$  ?
- (c) Prove that  $\sim$  is an equivalence relation.
- (d) List five elements from each of the following equivalence classes:  
 $\overline{(1, 1)}$ ,  $\overline{(0, 2)}$ ,  $\overline{(2, 3)}$ .

5. (Constructing the integers from the natural numbers) Let  $S = \mathbb{N} \times \mathbb{N}$ . Define the relation  $\sim$  on  $S$  where  $(a, b) \sim (c, d)$  if and only if  $a+d = b+c$ .

- (a) Is  $(3, 6) \sim (7, 10)$  ?
- (b) Is  $(1, 1) \sim (3, 5)$  ?
- (c) Prove that  $\sim$  is an equivalence relation.
- (d) List five elements from each of the following equivalence classes:  
 $\overline{(1, 1)}$ ,  $\overline{(1, 2)}$ ,  $\overline{(5, 12)}$ .

6. Let  $S = \mathbb{Z}$ . Define the relation  $\sim$  on  $S$  where  $x \sim y$  if and only if  $3x - 5y$  is even. Prove that  $\sim$  is an equivalence relation on  $S$ .